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FURTHER DEVELOPMENTS OF A MULTIFRACTAL MODEL OF ATMOSPHERIC TURBULENCE

BY

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Abstract. This paper present a multifractal interpretation of turbulent atmospheric entities, considering them a system whose dynamics are manifested on continuous yet non-differentiable multifractal curves. By bringing forth theoretical considerations regarding multifractal structures through non-differentiable functions in the form of an adaptation of scale relativity theory, the minimal vortex of an instance of turbulent flow is considered. This then leads to a general equation for the non-differentiable vortex itself with its component velocity fields – all of which are plotted and studied.

Keywords: turbulence; multifractal; non-differentiable; vortex.

1. Introduction

Atmospheric physics, as a distant relative to many of the fields of mechanics and dynamics, is supported by the fact that the principles of determinism apply to it; however, these principles do not have to imply a kind of cyclical behaviour in atmosphere dynamics. Considering most analytical

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considerations towards the atmosphere in general, it appears that predictability is a principal attribute of its afferent dynamics. However, once the techniques of nonlinear analysis and chaos theory have been developed, it is found that the reductionist analysis method, which has formed the basis of our understanding of the atmosphere, does not have a large range of applicability and that potentially-unlimited predictability cannot be linked the atmosphere; indeed, only through simplifying the description of the atmosphere through linear analysis can one arrive at unlimited predictability (Badii, 1997; Hou *et al*., 2009; Deville and Gatski, 2012).

First of all, there is no reason not to suppose that this non-linear nature of atmospheric phenomena is structural and functional, and various interactions between atmospheric entities certainly show connections and conditionings of the types macroscopic-microscopic, global-local, collective-individual, and many others. In such a theoretical framework, non-differentiable implementations for atmospheric dynamics are continuously discussed (Badii, 1997; Hou *et al*., 2009; Deville and Gatski, 2012). Non-differentiability implies fractality: continuously self-similar structures for which one must employ scale-dependent laws. Many models used for the description of atmospheric dynamics are founded on the hypothesis that variables describing it happen to be differentiable (Hou *et al*., 2009); but, as previously mentioned, integrability and differentiability are no longer fully applicable here. These procedures, in general, however, cannot function properly when describing nonlinear and chaotic processes, which is usually the case in atmospheric phenomena (Deville and Gatski, 2012).

In order to describe atmospheric dynamics in this new manner while also using certain differentiable mathematical methods, we have to bring forth the notion of scale resolution while expressing these variables and equations govern these atmospheric dynamics (Nottale, 2011; Agop and Păun, 2017). This then implies that any and all variables that are classically-dependent just on spatial and temporal coordinates depend, in a “non-differentiable manner”, on the resolution of their respective scales; or, instead of functioning with just one variable contained in a non-differentiable function, we shall operate just with function approximations found by averaging them on various scale resolutions. Thus, all variables that appear in the description of atmospheric dynamics shall work as the limit of a “family of mathematical functions”, being non-differentiable for null-scale resolutions and differentiable in other cases (Nottale, 2011; Agop and Păun, 2017). Such a manner of describing atmospheric dynamics implies developing a novel formalism and theory functional for such structures whose laws of dynamics, invariant to any spatial and temporal transformations become integrated with scale laws invariant to the transformations of scale resolutions.

2. Multifractal Model. Results and Discussions

Much of the theory behind the construction and reasoning of the associated mathematical and physical formalism can be found in one of our recent papers (Roșu *et al*., 2020); for now, this new structure could be attained if one considers that the differential of the spatial coordinate of the atmospheric multifractal structure $d_{\pm}X^{i}(t,dt)$ is expressed as the sum of the two differentials. One of them is scale resolution independent, $d_{\pm}x^{i}(t)$, and the other one is scale resolution dependent, $d_{\pm}\xi^{i}(t,dt)$ in the form:

$$d_{\pm}X^{i}(t,dt) = d_{\pm}x^{i}(t) + d_{\pm}\xi^{i}(t,dt) \tag{1}$$

The "+" sign corresponds to forwards processes of the atmosphere dynamics, while the "–" sign corresponds to backwards ones. Then, it is possible to state that the non-differentiable part of the spatial coordinate satisfies the multi-fractal equation (Agop and Păun, 2017):

$$d_{\pm}\xi^{i}(t,dt) = \lambda^{i}(dt)^{\left[2/f(\alpha)\right]-1}, \alpha = \alpha(D_F) \tag{2}$$

where λ^{i} are coefficients associated to differential-nondifferential transition, $f(\alpha)$ is the singularity spectrum of order α, and $\alpha(x)$ is the singularity (Hou *et al*., 2009). The singularity spectrum is defined:

$$f(\alpha) = D_F\{x, \alpha(x) = h\} \tag{3}$$

Of course, judging by the definition of the multifractal itself, the fractal dimension alone is not enough to characterize the atmospheric multifractal: this is why the previous equation states that the non-differentiable part of the spatial coordinate satisfies a relation where a singularity spectrum $f(\alpha)$ is found. This singularity spectrum denotes the dimension not of an entire atmospheric multifractal curve necessarily, however it describes a spectrum of dimensions that groups of points found in this curve might have, depending on whether or not the same Hölder exponent can be used for them. It is implied then, although it is not specified necessarily, that the smallest possible atmospheric multifractal, the minimal multifractal vortex, will be described by one Hölder exponent – then, indeed, $f(\alpha)$ becomes wholly equivalent to D_F.

It can be quite difficult to arrive at any analytical solutions for the equation system of the velocities described by this theory while considering nonlinearity created by non-differentiable convection and non-differentiable dissipation (Roșu *et al.*, 2020). However, it is possible to arrive at a solution in the case of plane symmetry of the dynamics of our given non-differentiable atmospheric turbulence unit. Thus, we consider the equation system of fractal hydrodynamics at a nondiferentiable scale resolution for the stationary case and at plane symmetry as:

$$u\partial_x u + v\partial_y u = \sigma\partial_{yy}^2 u \tag{4}$$

$$\partial_x u + \partial_y v = 0 \tag{5}$$

where:

$$u = u(x,y), v = v(x,y), \sigma = \lambda(dt)^{\left(\frac{2}{f(\alpha)}\right)-1} \tag{6}$$

In the upper relations, u and v are the components of the wind velocity dependent on the scale resolution dt, λ is a constant associated with the fractal-multifractal transition, $f(\alpha)$ is singularity spectrum of order α, and α is the singularity index of the movement curves of the atmospheric entities. Eq. (4) corresponds to the conservation law of the specific impulse at nondiferentiable scale resolutions, and Eq. (5) corresponds to the conservation law of state density at the nondiferentiable scale resolution (*i.e.* the incompressibility of the atmosphere at these scales). Now by introducing the adimensional variables:

$$\xi = \frac{x}{x_0}, \eta = \frac{y}{y_0}, U = \frac{u}{u_0}, V = \frac{v}{v_0} \tag{7}$$

With the property that:

$$u_0 y_0 = v_0 x_0 \tag{8}$$

The system in Eqs. (4) and (5) can be rewritten as:

$$U\partial_\xi U + V\partial_\eta U = \nu\partial_{\eta\eta}^2 U \tag{9}$$

$$\partial_\xi U + \partial_\eta V = 0 \tag{10}$$

where:

$$\nu = \frac{\sigma}{\sigma_0} = \frac{\sigma}{y_0 v_0} = \frac{\sigma x_0}{{y_0}^2 u_0} \tag{11}$$

In the previous equations, x_0 and y_0 are specific lengths, u_0 and v_0 are specific velocities and ν is the fractality degree (a measure of the fractality of the movement curves of the atmospheric entities), all of these quantifying the atmospheric characteristics at nondiferentiable scale resolutions. By adding the restrictions:

$$\lim_{\eta\to 0} V(\xi,\eta) = 0,\ \lim_{\eta\to 0}\frac{\partial U}{\partial \eta} = 0,\ \lim_{\eta\to\infty} U(\xi,\eta) = 0\,, \tag{12a}$$

$$q = q_0 \int_{-\infty}^{+\infty} U^2 d\eta = ct.,q_0 = \rho\frac{{u_0}^2}{y_0} \tag{12b}$$

Following the demonstration found in the previously-mentioned paper, the velocity field of the atmospheric entities at nondiferentiable scale resolutions is given by the relations:

$$U = \frac{1.5}{(\nu\xi\,)^{\frac{1}{3}}} sech^2\left[\frac{0.5\eta}{(\nu\xi\,)^{\frac{2}{3}}}\right] \tag{13}$$

$$V = \frac{1.9}{(\nu\xi\,)^{\frac{1}{3}}}\left\{\frac{\eta}{(\nu\xi\,)^{\frac{2}{3}}} sech^2\left[\frac{0.5\eta}{(\nu\xi\,)^{\frac{2}{3}}}\right] - tanh\left[\frac{0.5\eta}{(\nu\xi\,)^{\frac{2}{3}}}\right]\right\} \tag{14}$$

Thus, the velocity fields of the atmospheric entities at nondiferentiable scale resolutions are obtained, and represented in Figs. 1-8.

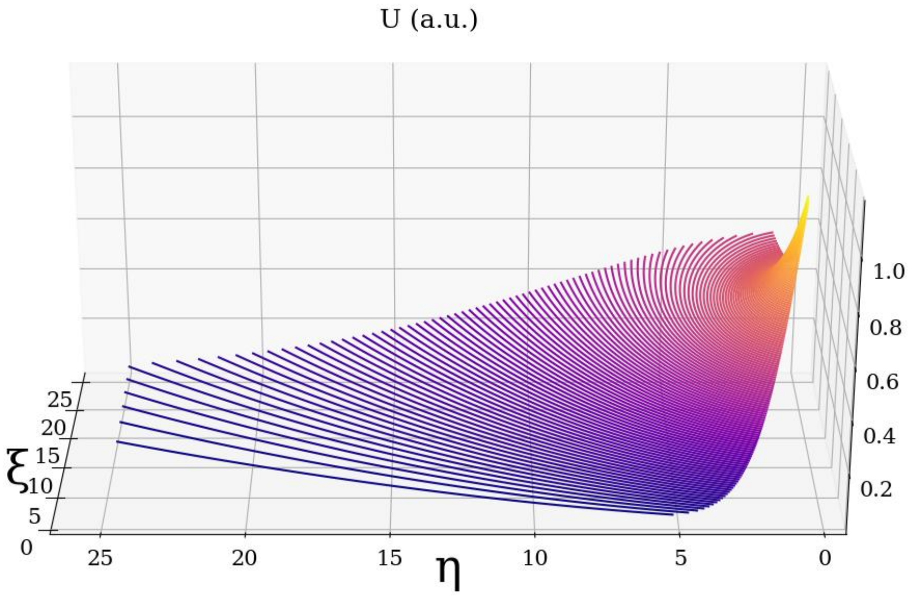

Fig. 1 – Normalized velocity field U of non-differentiable atmospheric multifractal; $\nu = 0.5$.

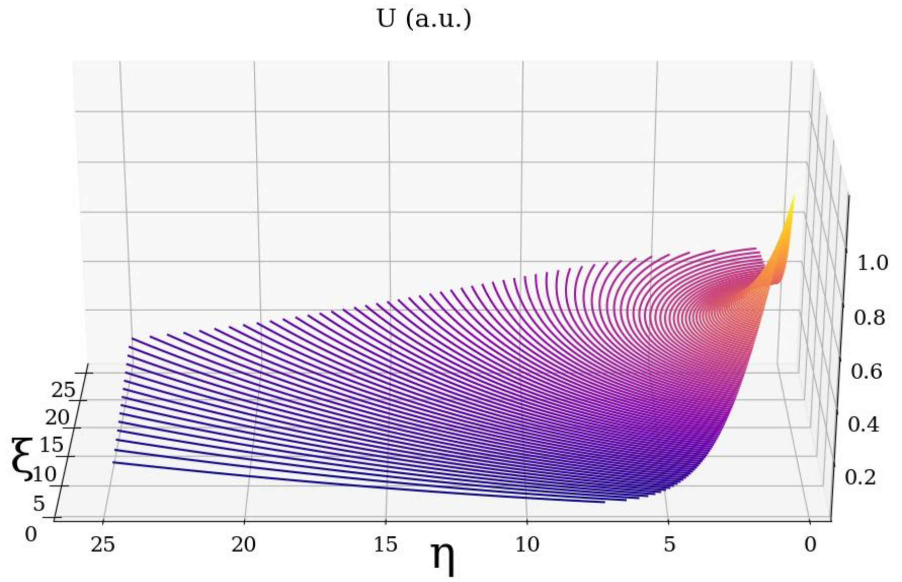

Fig. 2 – Normalized velocity field U of non-differentiable atmospheric multifractal; $\nu = 1$.

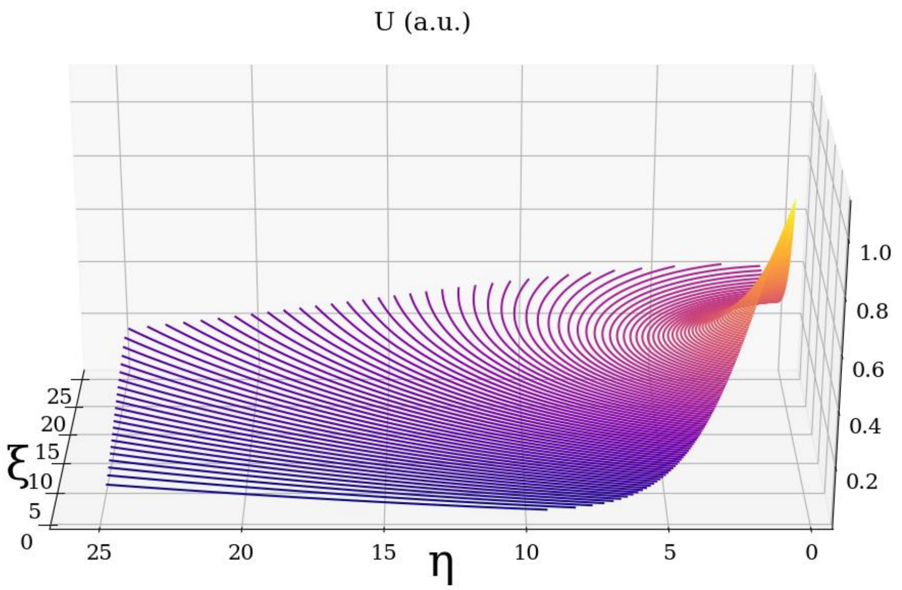

Fig. 3 – Normalized velocity field U of non-differentiable atmospheric multifractal; $\nu = 1.5$.

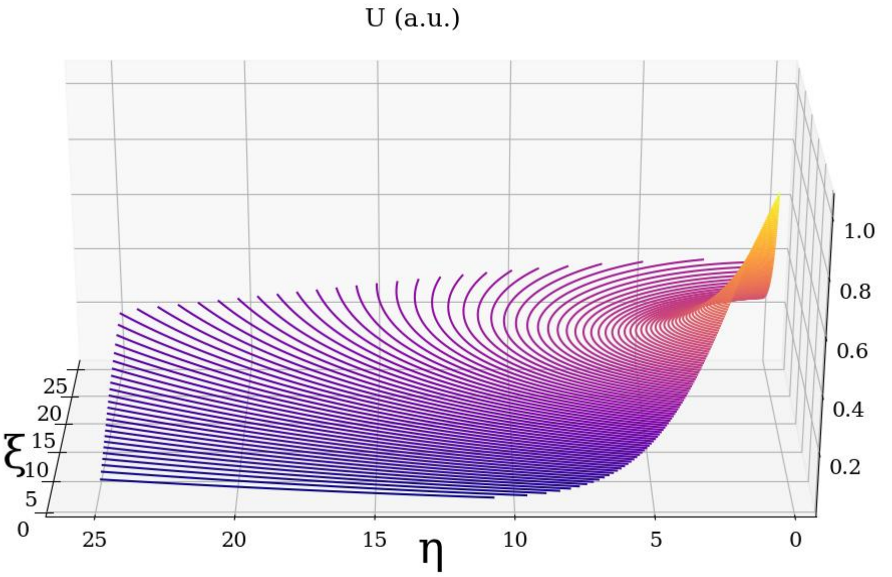

Fig. 4 – Normalized velocity field U of non-differentiable atmospheric multifractal; $\nu = 2$.

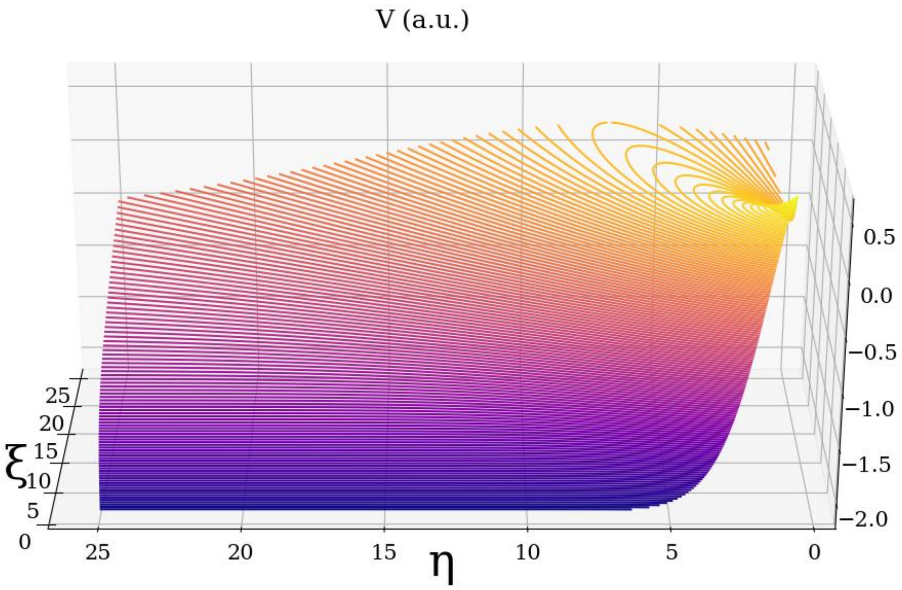

Fig. 5 – Normalized velocity field V of non-differentiable atmospheric multifractal; $\nu = 0.5$.

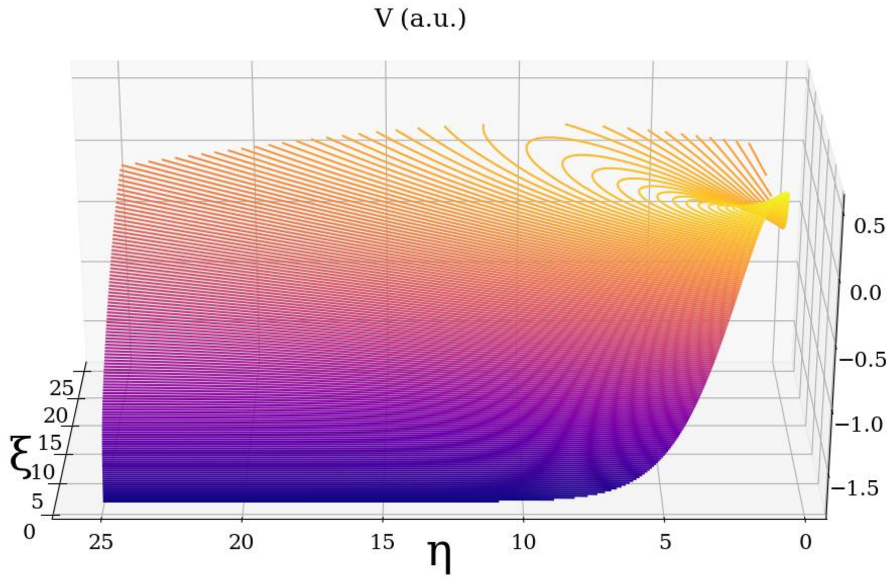

Fig. 6 – Normalized velocity field V of non-differentiable atmospheric multifractal; $\nu = 1$.

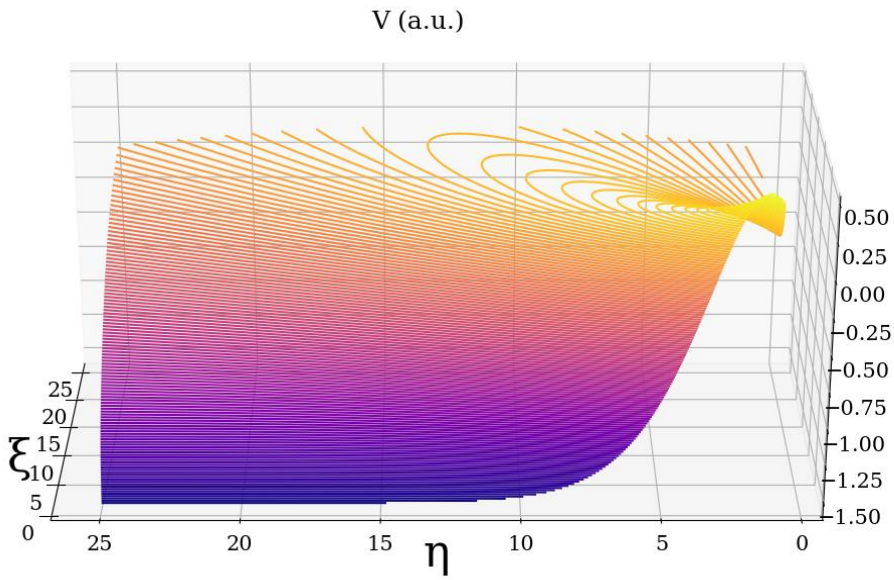

Fig. 7 – Normalized velocity field V of non-differentiable atmospheric multifractal; $\nu = 1.5$.

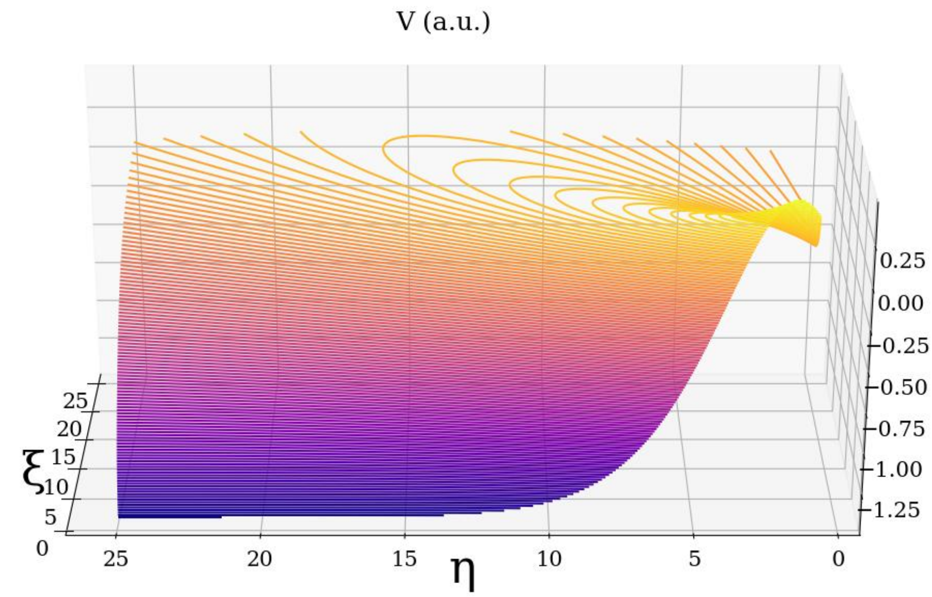

Fig. 8 – Normalized velocity field V of non-differentiable atmospheric multifractal; $\nu = 2$.

Now, through Eqs. (13) and (14), the vortex field at nondiferentiable scale resolutions is introduced:

$$\Omega = \left(\partial_\eta U - \partial_\xi V\right) = \frac{0.57\eta}{(\nu\xi)^2} + \frac{0.63\xi}{(\nu\xi)^{\frac{4}{3}}} tanh\left[\frac{0.5\eta}{(\nu\xi)^{\frac{2}{3}}}\right] + \frac{1.9\eta}{(\nu\xi)^2} sech^2\left[\frac{0.5\eta}{(\nu\xi)^{\frac{2}{3}}}\right] -$$

$$-\frac{0.57\eta}{(\nu\xi)^2} tanh^2\left[\frac{0.5\eta}{(\nu\xi)^{\frac{2}{3}}}\right] - \left[\frac{1.5}{\nu\xi} + \frac{1.4\eta}{\xi(\nu\xi)^{\frac{5}{3}}}\right] sech^2\left[\frac{0.5\eta}{(\nu\xi)^{\frac{2}{3}}}\right] tanh\left[\frac{0.5\eta}{(\nu\xi)^{\frac{2}{3}}}\right] \quad (15)$$

The vortex velocity field and virtual source of turbulence at nondiferentiable scale resolutions is shown in Figs. 9-12.

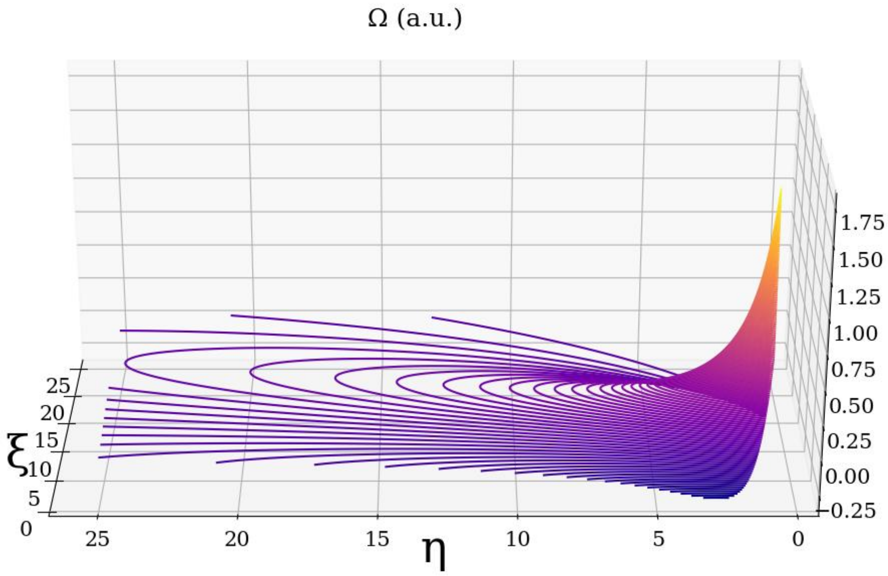

Fig. 9 – Normalized vortex Ω of non-differentiable atmospheric multifractal; $\nu = 0.5$.

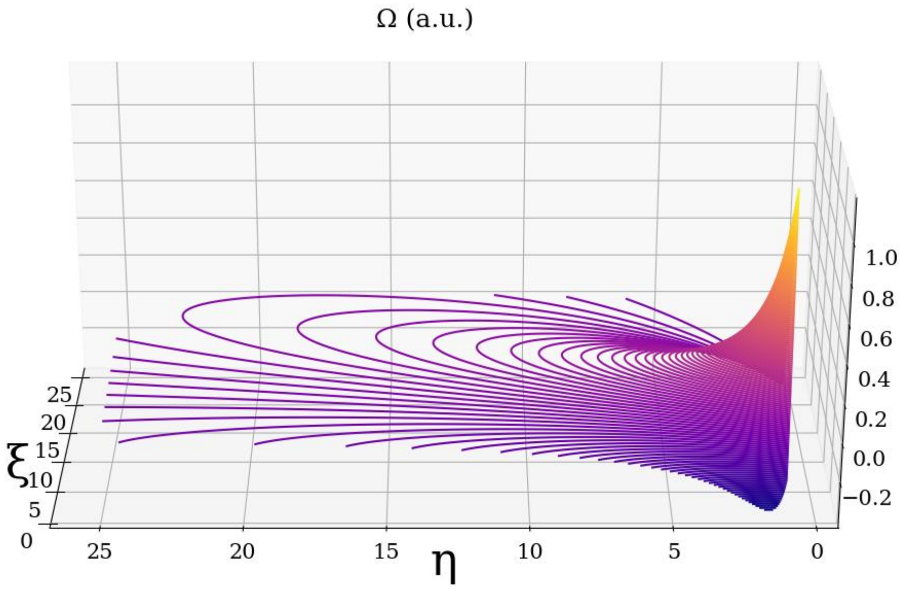

Fig. 10 – Normalized vortex Ω of non-differentiable atmospheric multifractal; $\nu = 1$.

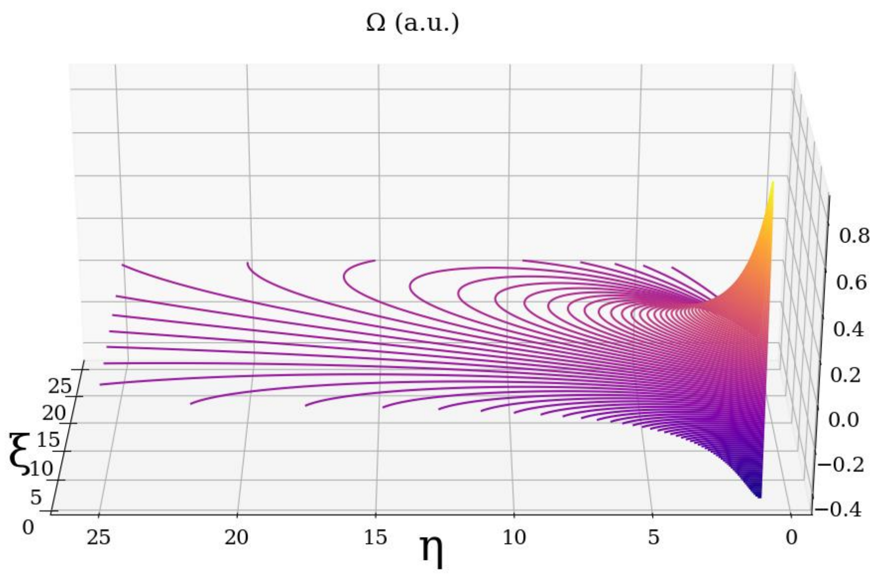

Fig. 11 – Normalized vortex Ω of non-differentiable atmospheric multifractal; $\nu = 1.5$.

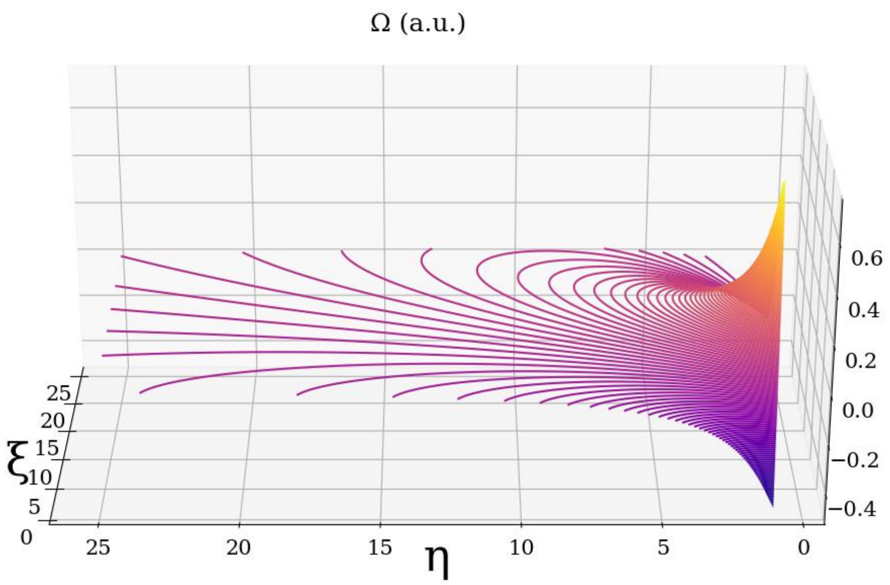

Fig. 12 – Normalized vortex Ω of non-differentiable atmospheric multifractal; $\nu = 2$.

Now, in order to quantify the dissipative character of the non-differentiable vortex, we may employ the following equation (Tatarski, 2016):

$$\varepsilon \cong 0.353{l_d}^2\Omega^3 \tag{16}$$

Thus, we can construct the field of the turbulent dissipation rate of the nonmanifest vortex (Figs. 13-16).

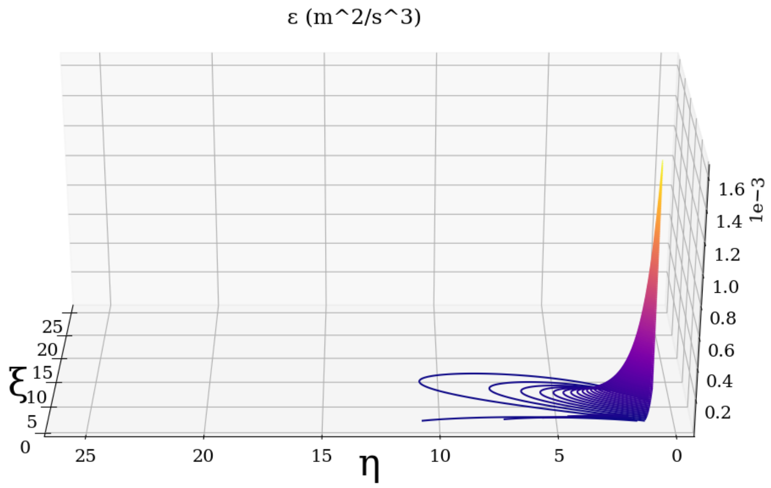

Fig. 13 – Turbulent dissipation field of minimal vortex of the non-differentiable atmospheric multifractal; ν = 0.5.

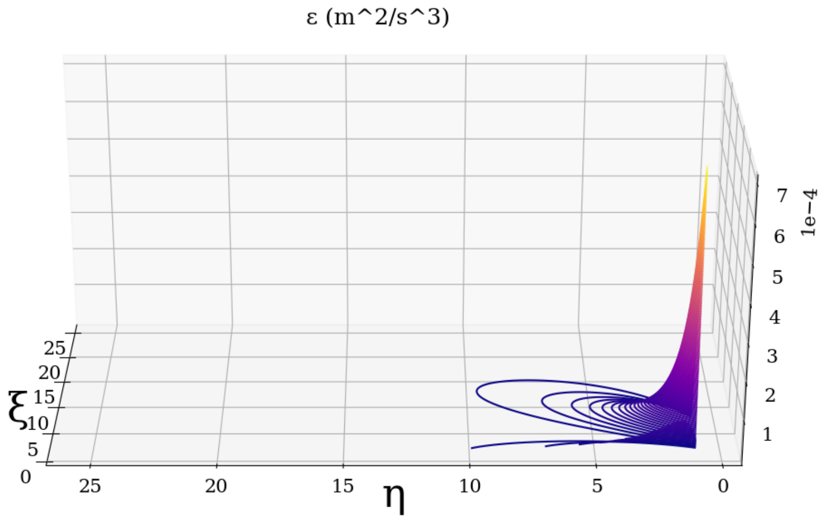

Fig. 14 – Turbulent dissipation field of minimal vortex of the non-differentiable atmospheric multifractal; ν = 1.

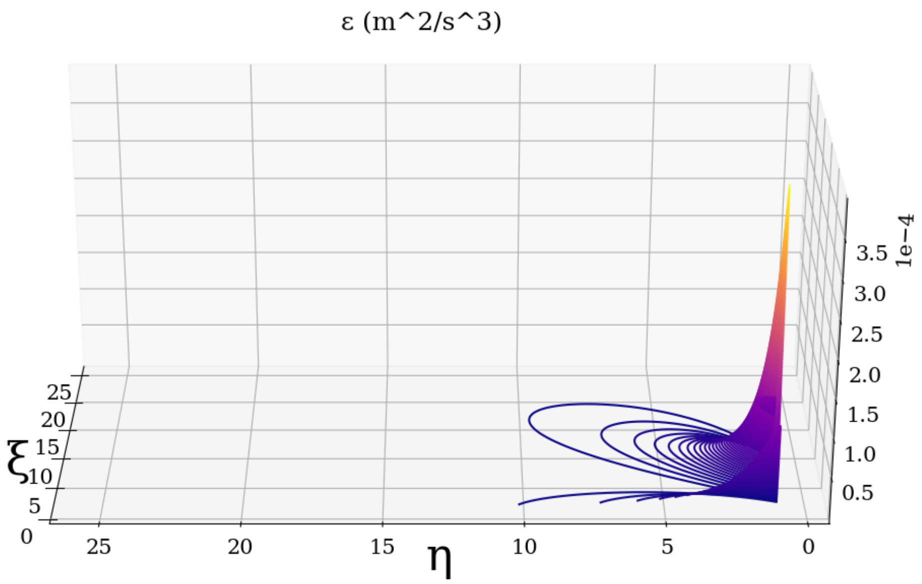

Fig. 15 – Turbulent dissipation field of minimal vortex of the non-differentiable atmospheric multifractal; $\nu = 1.5$.

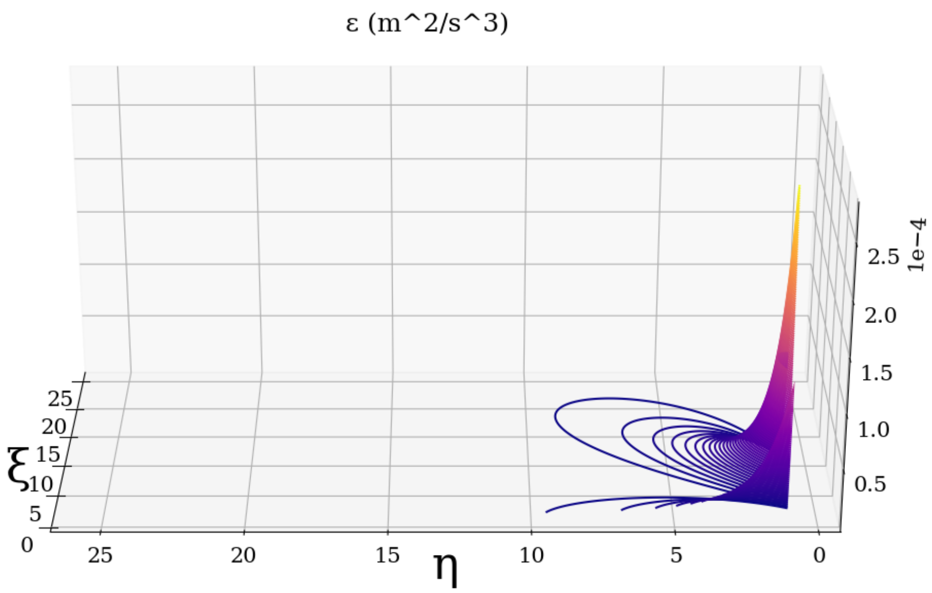

Fig. 16 – Turbulent dissipation field of minimal vortex of the non-differentiable atmospheric multifractal; $\nu = 2$.

The minimal vortex field becomes manifest and a real source of turbulence at diferentiable scale resolutions through coherence, or the autostructuring of minimal vortices in vortex streets. Indeed, by considering a two-dimensional non-differentiable and non-coherent fluid (which will then exhibit fractal-like qualities), its entities, which scale down towards minimal vortices, become structured as a vortex lattice of cnoidal stationary modes.

In terms of the results, the velocity fields of this multifractal minimal vortex and its dissipation field have been plotted in Figs. 1-12; the greatest velocity shown by the vortex seems to lie in the vicinity of its centre, and, as one would expect, turbulent dissipation is also strongest near the centre. Surprisingly, this structure shows a "downward-spiral" motion, where the maximum and minimum values are in the vicinity of each other, and the trajectory from one to the other appears to be the shortest right along the x-axis towards 0. We might interpret this to mean that it points to the way in which the vortex dissipates and the flow progresses; an "innate directionality" to the structure of these vortices regarding their velocity fields may give rise to small vortex-local gradients of fluid density for each of the vortices, which, if aligned, may produce vortex streets and lattices. Increasing our non-dimensional parameter ν shall produce a decidedly-smaller vortex minimum and a "less acute" dissipation field (slightly larger in proportion to the others, with smaller values) (Figs. 13-16), however, by increasing the parameter, we arrive at velocity fields that give a "sharper" maximal and a more spread-out minimal region in rotation. In any case, we have assumed that these simulated phenomena take place in calm ground-level conditions for a normal atmosphere.

3. Conclusion

In this article, the non-linear behavior and functionality of the atmosphere has been placed under a multifractal approach, by supposing that turbulent atmospheric flow can be likened to a complex system where both the functional and structural units exhibit dynamics on continuous yet non-differentiable trajectories. We have named this system the "atmospheric multifractal", and by formulating a multifractal interpretation, an analytic solution for planar symmetry for the velocity fields of the system has been obtained. The rotor of these newly-obtained normalized velocity fields is found to be a generalized multifractal vortex. Multiple instances of a non-dimensional parameter is used to represent the velocity fields and the vortex; this parameter is linked to their fractal dimensions, and the plotted results are then shown and discussed in relation to this connection.

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CONTRIBUȚII ADUSE UNUI MODEL MULTIFRACTAL AL TURBULENȚEI ATMOSFERICE

(Rezumat)

Acest articol prezintă o interpretare multifractală a entităților turbulenței atmosferice, considerându-le un sistem al cărui dinamici se manifestă pe curbe multifractale nediferențiabile, dar continue. Aducând considerații teoretice în legătură cu structuri multifractale prin funcții nediferențiabile sub forma unei adaptări a teoriei relativității de scară, se specifică vortexul minimal al unei instanțe de curgere turbulentă. Astfel se ajunge la o ecuație generală a vortexului nediferențiabil, compus din câmpurile de viteză aferente, care sunt analizate grafic pentru diverse grade de fractalitate.